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"On the Solution of the Differential Equations Resulting from the Separation of Laplace Equation in Various Coordinate Systems, by Means of Lie Series," by F.Cap and A.Schett.

I Introduction

The Helmholtz equation

$$\Delta \varrho + \varkappa^2 \varrho = 0 \qquad (I,1)$$

and the Laplace's equation

$$\Delta \varrho = 0 \qquad (I,2)$$

have a great significance in physics. There are many equations, important for physical and technical applications, which reduces to Helmholtz equation if time dependence is separated. These equations are i.a

1) The diffusion equation:

$$\Delta^2 \varrho = \frac{1}{h^2} \frac{\partial \varrho}{\partial t}$$

This type of equation appears, f.e., in heat conduction theory, diffusion theory and circulatory motion theory.

2) The wave equation:

$$\Delta^2 \varrho = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2}$$

3) The damped wave equation:

$$\Delta^2 \varrho = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2} + R \frac{\partial \varrho}{\partial t}$$

4) The transmission line equation:

$$\Delta^2 \varrho = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2} + R \frac{\partial \varrho}{\partial t} + S\varrho$$

5) The vector wave equation:

$$\Delta^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

The equations enumerated under 1) -5) describe quite generally the propagation of waves.

The <u>Laplace equation</u> occurs, f.e., in elasticity theory (stress problems, tortion problems, distortion problems, thermal elasticity problems a.s.o.), in potential theory and in potential flow problems. Concerning the separability it is well-known that these equations can be separated in special coordinate systems. One distinguishes R and S separability.

S-separability: If the assumption

$$\varrho = U_1(n_1) \cdot U_2(n_2) \cdot U_3(n_3) \tag{I,3}$$

permits the separation of the partial differential equations (1) and (2), respectively, into three ordinary differential equations, the equation is said to be simply separable or S-separable.

R-separability: If the assumption

$$\varrho = \frac{U_1(n_1)\ U_2(n_2)\ U_3(n_3)}{R(n_1,\ n_2,\ n_3)} \tag{I,4}$$

permits the separation of the partial differential equations (1) and (2), respectively, into three ordinary differential equations, and if R = const., the equation is said to be R-separable. The quantity R is defined in /1/.

No case is known in which the Helmholtz equation is R-separable, so the question that arises is merely wether the Laplace equation is R-separable in some coordinate systems. In the following table we list the R and S separability of the Laplace and Helmholtz equation, respectively in various coordinate systems. We restrict ourselve to the well-known 11 coordinate systems in which the Helmholtz equation is separable and the most important coordinate systems with regard to technical problems in which the Laplace Equation is R-separable.

In Table I S indicates S-separability

R " R- "

X " non-separability

TABLE I

Coordinate System	$\Delta\varrho + \varkappa^2\varrho = 0$		$\Delta\varrho = 0$	
	$\varrho(n_1 n_2 n_3)$	$\varrho(n_1 n_2)$	$\varrho(n_1 n_2 n_3)$	$\varrho(n_1 n_2)$
1. Rectangular Coordinates	S	S	S	S
2. Circular-Cylinder Coordinates	S	S	S	S
3. Elliptic-Cylinder Coordinates	S	S	S	S
4. Parabolic-Cylinder Coordinates	S	S	S	S
5. Spherical Coordinates	S	S	S	S
6. Prolate spheroidal Coordinates	S	S	S	S
7. Oblate spheroidal Coordinates	S	S	S	S
8. Parabolic Coordinates	S	S	S	S
9. Conical Coordinates	S	S	S	S
10. Ellipsoidal Coordinates	S	S	S	S
11. Paraboloidal Coordinates	S	S	S	S
12. Tangent-Cylinder Coordinates	X	X	X	S
13. Cardioid-Cylinder Coordinates	X	X	X	S

TABLE I	$\Delta \varrho + \varkappa^2 \varrho = 0$		$\Delta \varrho = 0$	
Coordinate System	$\varrho(n_1 n_2 n_3)$	$\varrho(n_1 n_2)$	$\varrho(n_1 n_2 n_3)$	$\varrho(n_1 n_2)$
14. Hyperbolic-Cylinder Coordinates	X	X	X	S
15. Rose Coordinates	X	X	X	S
16. Cassian-Oval Coordinates	X	X	X	S
17. Inverse Cassian-Oval Coordinates	X	X	X	S
18. Bi-Cylindrical Coordinates	X	X	X	S
19. Maxwell-Cylinder Coordinates	X	X	X	S
20. Logarithmic-Cylinder Coordinates	X	X	X	S
21. ln tang Cylinder Coordinates	X	X	X	S
22. ln cosh Cylinder Coordinates	X	X	X	S
23. sn-Cylinder Coordinates	X	X	X	S
24. cn-Cylinder Coordinates	X	X	X	S
25. Inverse sn-Cylinder Coordinates	X	X	X	S

TABLE I

Coordinate Systems	$\Delta\varrho + \varkappa^2\varrho = 0$		$\Delta\varrho = 0$	
	$\varrho(n_1n_2n_3)$	$\varrho(n_1n_2)$	$\varrho(n_1n_2n_3)$	$\varrho(n_1n_2)$
26. ln sn-Cylinder Coordinates	X	X	X	S
27. ln cn-Cylinder Coordinates	X	X	X	S
28. Zeta Coordinates	X	X	X	S
29. Tangent Sphere Coordinates	X	X	R	R
30. Cardioid Coordinates	X	X	R	R
31. Bispherical "	X	X	R	R
32. Toroidal "	X	X	R	R
33. Inverse prolate spheroidal Coordinates	X	X	R	R
34. Inverse oblate spheroidal Coordinates	X	X	R	R
35. Bi-Cyclide Coordinates	X	X	R	R
36. Flat-Ring Cyclide Coordinates	X	X	R	R
37. Disk-Cyclide Coordinates	X	X	R	R
38. Cap-Cyclide Coordinates	X	X	R	R

All differential equations which result from a separation of the Helmholtz equation are special cases of the Bôcher equation. Initial value problems of this general equation were solved by Lie Series in Rep.2 and Rep.3 under the Contract N 6 R 52-046-001. Special cases were treated in Rep.7 and Rep.8 under the same Contract and in the Monograph entitled: Solution of Ordinary Differential Equations by Means of Lie Series, by F.Cap, D.Floriani, W.Groebner, A.Schett and J.Weil, published by NASA.

The differential equations which result from a separation of the Laplace equation are also contained in the Bôcher equation.

It means that for initial value problems these ordinary differential equations are solved too. Here we enumerate for the sake of completeness the types of the differential equations resulting from a separation of Laplace's equation in various coordinate systems. Concerning the solution of different types we refer to earlier report under the Contract NGR 52-046-001, if the equation is treated already or shall solve the equation for initial value problems, if the equation is not investigated in earlier reports already. We emphasize, Lie series can only be used to representate functions in regular domains.

II Types of Differential Equations Resulting from a Separation of Laplace Equation in some important Coordinate Systems; $\varrho = \varrho(n_1 n_2 n_3)$.

Type I:

$$Z''(t) - cZ(t) = 0 \qquad (II,1)$$

c being a constant. This type appears in:

rectangular coordinates,
circular cylinder coordinates,
elliptic-cylinder coordinates,
parabolic-cylinder coordinates,
spherical coordinates
prolate spheroidal coordinates,
oblate spheroidal coordinates,
parabolic coordinates,
tangent-sphere coordinates,
cardioid coordinates,
bispherical coordinates,
toroidal coordinates,
inverse prolate spheroidal coord.
inverse oblate spheroidal coord.
bi-cyclide coordinates,
flat-ring cyclide coordinates,
disk-cyclide coordinates,
cap-cyclide coordinates.

This type is already treated in Rep.7.

Type II:

$$Z''(t) + \frac{a}{t}Z'(t) - \left(\frac{b}{t^2} + c\right) Z(t) = 0 \qquad \text{(II,2)}$$

a, b , c being constants. This equation appears among the equation of

circular cylinder coord.(a = 1),
parabolic coordinates,
tangent-sphere coordinates,
spherical coordinates
conical coordinates,
cardioid coordinates.

Eq.(II,2) is already treated in Rep.7.

Type III:

$$Z''(t) + (a + bt^2)\, Z(t) = 0 \qquad \text{(II,3)}$$

Eq.(II,3) appears among the equations of parabolic cylinder coordinates. For the solution of this type see Rep.7.

Type IV:

$$Z''(t) - (\alpha_2 + \alpha_3 a^2 \cosh^2 t)\, Z(t) = 0 \qquad \text{(II,4)}$$

α_2, α_3, a being constants. This equation results from a separation of the Laplace equation in elliptic cylinder coordinates. For solving this equation see Rep.8.

Type V:

$$Z''(t) + \coth t\, Z'(t) + (\varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t})\, Z(t) = 0 \qquad (II,5)$$

$\varkappa$, a, α_2, α_3 being constants.

This equation appears among the equations in

prolate spheroidal (a=0) coord., toroidal (a=0) coordinates,

inverse prolate spheroidal (a=0) coordinates

Type VI:

$$Z''(t) + \cot t\, Z'(t) + (\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t}\, Z(t) = 0 \qquad (II,6)$$

$\varkappa$, a, α_2, α_3 being constants.

This equation results from:

spherical (a = 0) coordinates,

prolate spheroidal (a = 0) coordinates

oblate spheroidal (a = 0) coordinates,

bispherical (a = 0) coordinates,

inverse prolate spheroidal (a = 0) coordinates,

inverse oblate spheroidal (a = 0) coordinates.

Eq.(II,6) was investigated in Rep.8 already.

Type VIII:

$$Z''(t) + \text{tangh}\, t\, Z'(t) + (\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t})Z(t) = 0 \qquad (II,7)$$

$\varkappa$, a, α_2, α_3 being constants. This equation results from:

oblate spheroidal (a = 0) coordinates,

inverse oblate spheroidal (a = 0) coordinates.

The solution is given in Rep.8.

Type VIII:

$$Z''(t) + \frac{t(2t^2-(b^2+c^2))}{(t^2-b^2)(t^2-c^2)} Z'(t) + \frac{(\varkappa^2 t^4+\alpha_3 t^2+\alpha_2)}{(t^2-b^2)(t^2-c^2)} Z(t) = 0 \qquad \text{(II,8)}$$

b, c, $\varkappa$, α_2, α_3 being constants.

This equation appears among the equations in

conical coordinates,

ellipsoidal coordinates.

For solving Eq.(II,8) see Rep.8.

Type IX:

$$Z''(t) + \frac{1}{2}\frac{(2t-(b+c))}{(t-b)(t-c)} Z'(t) + \frac{\varkappa^2 t^2+\alpha_3 t-\alpha_2}{(t-b)(t-c)} Z(t) = 0 \qquad \text{(II,9)}$$

b, c, $\varkappa$, α_2, α_3 being constants.

Eq.(II,9) results from a separation of the Laplace equation in paraboloidal coordinates. The solution for initial value problems is given in Rep.8.

Type X:

$$Z''(t) + \left[\frac{1}{2}\left(\frac{1}{t-a_1} + \frac{2}{t-a_2} + \frac{3}{t-a_3}\right)\right] Z'(t) +$$

$$+ \frac{1}{4}\left[\frac{b_0+b_1 t+b_2 t^2+b_3 t^3}{(t-a_1)(t-a_2)^2(t-a_3)^2}\right] Z(t) = 0 \qquad \text{(II,10)}$$

where k, α_2, α_3, a_i, b_j being constants ($i = 1,2,3$; $j = 0,1,2,3$)

The solution functions of Eq.(II,10) are <u>Heine functions</u> /1/.

Let $a_1 = 0$, $a_2 = 1$, $a_3 = 1/k^2$,

$$b_o = -\frac{\alpha_2}{k^2}$$

$$b_1 = (\alpha_2+2) + \frac{\alpha_2}{k^2} - \frac{\alpha_3 k'^4}{k^2}$$

$$b_2 = (\alpha_2+2) + 2k^2$$

$$b_3 = 2k^4$$

$0 < k^2 < 1$

$0 < k'^2 < 1$

and

$t = sn^2\zeta$

then one obtains the equation

$$Z''(\zeta) - \frac{sn\zeta(dn^2\zeta + k^2 cn^2\zeta)}{cn\zeta\; dn\zeta} Z'(\zeta) + \left[2k^2 sn^2\zeta - \alpha_2 - \alpha_3 \frac{k'^4 sn^2\zeta}{cn^2\zeta\; dn^2\zeta}\right] Z(\zeta) = 0 \qquad (II,10a)$$

where the Jacobi elliptic functions:

sn - sinus amplitudinis

cn - cosinus amplitudinis

dn - delta amplitudinis

Eq.(II,10a) results from a separation of the Laplace equation in bi-cyclide coordinates. Obviously the solution functions of Eq.(II, 10a) are Heine functions /1/.

Let $a_1 = 0$, $a_2 = 1$, $a_3 = k^2$

$b_o = -\alpha_2 k^2$

$b_1 = (\alpha_2 - \alpha_3) + k^2(\alpha_2 + 2)$

$b_2 = -(\alpha_2 + 2) - 2k^2$

$b_3 = 2$

and

$t = dn^2\zeta$,

we obtain from Eq.(II,10) the equation

$$Z''(\zeta) + \frac{dn\zeta(cn^2\zeta - sn^2\zeta)}{sn\zeta\, cn\zeta} Z'(\zeta) + \left[-2dn^2\zeta + \alpha_2 + \alpha_3 \frac{dn^2\zeta}{sn^2\zeta\, cn^2\zeta}\right] Z(\zeta) = 0 \qquad (II,10b)$$

which is again solved by <u>Heine functions</u> /1/.

Eq.(II,10b) appears among the equations which one obtains by separation of Laplace's equation in <u>bi-cyclide</u> coordinates. The general solution of Eq.(II,10) is given by

$$Z(t) = A\mathcal{H}_1(t) + B\mathcal{H}_2(t) \qquad (II,11)$$

where $\mathcal{H}_i$ $(i = 1,2)$ are Heine functions.

For <u>regular</u> domains we can solve Eq.(II,10) by Lie series. As Eqs.(II,10a), (II,10b) are special cases of Eq.(II,10) we have only to treat Eq.(II,10).

The solution representation for initial value problems is given in Rep.2 and Rep.3 by Eq.(14) and Eq.(44), respectively.

Eq.(14) in Rep.2 reads:

$$Z(t) = \sum_{0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} \cdot$$

$$\left(f_1^{(\varrho)}(z_o)\, D^{\nu-1-\varrho} z_1 + f_2^{(\varrho)}(z_o)\, D^{\nu-2-\varrho} z_1 \right. + z_1 + t z_2 \qquad \text{(II,12)}$$

The operator D is given by:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} +$$

$$\left(\frac{1}{2} \left[\frac{1}{t-a_1} + \frac{2}{t-a_2} + \frac{2}{t-a_3} \right] z_2 + \right.$$

$$\left. - \frac{1}{4} \left[\frac{b_o + b_1 t + b_2 t^2 + b_3 t^3}{(t-a_1)(t-a_2)^2 (t-a_3)^2} \right] \right) \frac{\partial}{\partial z_2} \qquad \text{(II,13)}$$

$t \neq a_1,\ t \neq a_2,\ t \neq a_3$

$$f_1(t) = -\left(\frac{1}{2(t-a_1)} + \frac{1}{t-a_2} + \frac{1}{t-a_3}\right)$$

$$f_2(t) = -\left(\frac{b_o + b_1 t + b_2 t^2 + b_3 t^3}{4(t-a_1)(t-a_2)^2 (t-a_3)^2}\right) =$$

$$= \frac{A_1}{t-a_1} + \frac{A_2}{(t-a_2)^2} + \frac{A_3}{t-a_2} + \frac{A_4}{(t-a_3)^2} + \frac{A_5}{t-a_3}$$

$$f_1^{(\varrho)}(t) = \frac{(-1)^{\varrho+1} \varrho!}{2(t-a_1)^{\varrho+1}} + \frac{(-1)^{\varrho+1} \varrho!}{(t-a_2)^{\varrho+1}} + \frac{(-1)^{\varrho+1} \varrho!}{(t-a_3)^{\varrho+1}}$$

$$f_2^{(\varrho)}(t) = \frac{A_1 (-1)^\varrho \varrho!}{(t-a_1)^{\varrho+1}} + \frac{A_2 (-1)^\varrho (\varrho+1)!}{(t-a_2)^{\varrho+2}} + \frac{A_3 (-1)^\varrho \varrho!}{(t-a_2)^{\varrho+1}} +$$

$$+ \frac{A_4 (-1)^\varrho (\varrho+1)!}{(t-a_3)^{\varrho+2}} + \frac{A_5 (-1)^\varrho (\varrho)!}{(t-a_3)^{\varrho+1}} \qquad \text{(II,14)}$$

Eqs. (II,12), (II,13), (II,14) solve equation (II,10) for <u>regular</u> domains.

The solution representation Eq.(44) in Rep. 3 reads:

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (T^{-1})^T \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} T^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} +$$

$$+ \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]_{\bar{a}} d\tau \qquad \text{(II,15)}$$

The integral can be evaluated by an iterative method according to /2/. The symbol $_{\bar{a}}$ added after the bracket is to indicate that after application of the D-operators z_1, z_2 have to be replaced by $e^{t\ D_1} z_1$ and $e^{t\ D_1} z_2$, respectively. λ_1, λ_2, T and D_2 in Eq. (II,15) are given by the relations:

$$\lambda_{1,2} = \frac{f_1}{2} \pm \sqrt{\frac{f_1^2}{4} + f_2}$$

$$T = \begin{pmatrix} f_2 & f_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \qquad \text{(II,16)}$$

$$D_2 = \frac{\partial}{\partial z_0}$$

Eqs.(II,15), (II,16) solve Eq.(II,10). If the initial values $Z(t=t_0)$ and $Z'(t=t_0)$ are given, the solution of Eq.(II,10) can be evaluated for regular domains.

The values $Z(t=t_0)$ and $Z'(t=t_0)$ can be looked up in tables.

The question arises how we can compute the <u>Heine functions</u> by Lie series representation Eqs.(II,12), (II,15).

The general solution of Eq.(II,10) is given by

$$Z(t) = A\mathcal{H}_1(t) + B\,\mathcal{H}_2(t)$$

A and B being arbitrary constants, $\mathcal{H}_1$, $\mathcal{H}_2$ are Heine functions. The solution and its derivative is given by:

$$Z(t) = A\mathcal{H}_1 + B\,\mathcal{H}_2 = Z_1(t)$$

$$Z'(t) = A\,\mathcal{H}'_1 + B\mathcal{H}'_2 = Z_2(t)$$

Without restriction of generality we may choose:

$$Z(t=t_o) = \mathcal{H}_1(t=t_o) = z_1$$

$$Z'(t=t_o) = \mathcal{H}_2(t=t_o) = z_2$$

i.e. we have put A = 1 and B = 0. Further the equations are valid.

$$Z(t) = \mathcal{H}_1(t) = \sum_0^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1$$

$$Z'(t) = \mathcal{H}_2(t) = \sum_1^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 \qquad \text{(II,A)}$$

For numerical evaluation of $\mathcal{H}_1$ we expand Z(t) in the neighborhood of $t = t_o$ and choose a step size of Δt. As t increases more terms $\frac{t^\nu D^\nu}{\nu!} z_1$ have to be calculated if the accuracy is prescribed. Since the computers have a limited numerical range, only a limited number of terms $\frac{t^\nu D^\nu}{\nu!} z_1$ can be calculated. Consequently, we expand the functions Z(t) at $t = t_o$ and using a certain step size Δt we calculate the function $\mathcal{H}_1$ in the region

t_o, t_1 . At t_1 the function $\mathcal{X}_1$ will be expanded again. Continuing this method, we can compute $\mathcal{X}_1$ and $\mathcal{X}'_1$ for regular domains.
In an analogous way one may calculate the function $\mathcal{X}_2$ by means of Lie series. Concerning the important problem of error estimation of the solution representations Eq.(II,12) and Eq.(II,15) we refer to already published papers /2, 3/.
G.MAESS treats in paper /3/ an error estimation, which may perhaps be used for numerical computation of Eq.(II,12). As we have never used this method, we cannot decide, wether this error estimation is suitable for numerical evaluation of Eq.(II,12).
H.KNAPP discusses in paper /2/ the error estimation of the representation Eq.(II,15). The usefulness of this method was already proved by numerical calculations /2/.
Whether Eq.(II,12) or Eq.(II,15) is more advantageous for computing the solutions can only be decided by help of a computer.

Type XI:

$$Z''(t) + \frac{1}{2}\left[\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3}\right] Z'(t) + \\ + \frac{1}{4}\left[\frac{b_o+b_1t+b_2t^2}{(t-a_1)(t-a_2)(t-a_3)^2}\right] Z(t) = 0 \qquad \text{(II,18)}$$

where, a_1, a_2, a_3, b_o, b_1, b_2, b_3 being constants. The solution functions of Eq.(II,18) are <u>Wangerin functions</u>.

If $a_1 = 1$, $a_2 = 1/k^2$, $a_3 = 0$

$$b_o = -\frac{\alpha_3}{k^2}$$

$$b_1 = \frac{-\alpha_2}{k^2} \qquad \text{(II,19)}$$

$$b_2 = 1-\alpha_3$$

and

$$t = \mathrm{sn}^2\zeta$$

one obtains from Eq.(II,18) the equation

$$Z''(\zeta) + \frac{\mathrm{cn}\,\zeta \,.\, \mathrm{dn}\,\zeta}{\mathrm{sn}\,\zeta} Z''(\zeta) +$$

$$+ \left[k^2\mathrm{sn}^2\zeta - \alpha_2 - \alpha_3 (k^2\mathrm{sn}^2\zeta + \frac{1}{\mathrm{sn}^2\zeta}) \right] Z(\zeta) = 0 \qquad \text{(II,18a)}$$

k, α_2, α_3 being constants

cn cosinus amplitudinis

sn sinus "

dn delta "

This equation results from a separation of Laplace's equation in

flat-ring coordinates,

cap-cyclide coordinates.

By the transformation $t = \mathrm{cn}^2\zeta$ one obtains with Eqs.(II,19) and Eq.(II,18) the differential equation

$$Z''(\zeta) - \frac{\mathrm{sn}\zeta\,\mathrm{dn}\zeta}{\mathrm{cn}\,\zeta} Z'(\zeta) +$$

$$+ \left[k^2\mathrm{sn}^2\zeta - \alpha_2 + \alpha_3 (k^2\mathrm{cn}^2\zeta - \frac{k'^2}{\mathrm{cn}^2\zeta}) \right] Z(\zeta) = 0 \qquad \text{(II,18b)}$$

If $a_1 = 1$, $a_2 = -(\frac{k'}{k})^2$, $a_3 = 0$

$b_0 = (k'/k)^2$

$b_1 = (\alpha_2 - k^2)/k^2$

$b_2 = \alpha_3 - 1$

This equation appears among the separated Laplace equation of <u>disk-cyclide</u> coordinates.

If $a_1 = 1$, $a_2 = -(k'/k)^2$, $a_3 = 0$

$b_o = \alpha_3$

$b_1 = (k'^2 - \alpha_2)/k^2$

$b_2 = 1 - \alpha_3 (k'/k)^2$

and

$$t = cn^2 \zeta$$

one obtains from (II,18) the equation

$$Z''(\zeta) - \frac{sn\zeta dn\zeta}{cn\zeta} Z'(\zeta) +$$

$$+ \left[-dn^2\zeta + \alpha_2 + \alpha_3 (k'^2 cn^2\zeta - \frac{k^2}{cn^2\zeta}) \right] Z = 0 \qquad \text{(II,18c)}$$

This equation appears among the separated Laplace equations in <u>disk-cyclide</u> coordinates.

If $a_1 = 1$, $a_2 = k^2$, $a_3 = 0$

$b_o = -\alpha_3 k^2$

$b_1 = -a_2$

$b_3 = 1 - \alpha_3$

and

$$t = dn^2 \zeta$$

one obtains from the origin equation (II,18) the equation:

$$Z''(\zeta) - \frac{k'^2 sn\zeta cn\zeta}{dn\zeta} Z'(\zeta) +$$

$$+ \left[-dn^2\zeta + \alpha_2 + \alpha_3 (dn^2\zeta + \frac{k^2}{dn^2\zeta}) \right] Z(\zeta) = 0 \qquad \text{(II,18d)}$$

This equation results from a separation of the Laplace's equation in

flat-ring coordinates,

cap-cyclide coordinates.

Eqs. (II,18a), (II,18b), (II,18c) and (II,18d) are solved by Wangerin functions.

The above enumerated equations are special cases of Eq.(II,18). Therefore we have only to solve Eq.(II,18).

For regular domains we can representate the solution by Lie series. For this case the solution is given by (II,12) and (II,15), respectively. The operator D reads:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + \Big(- \frac{1}{2} \left[\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3} \right] z_2 - \\ - \frac{1}{4} \left[\frac{b_0+b_1 t+b_2 t^2}{(t-a_1)(t-a_2)(t-a_3)^2} \right] z_1 \Big) \frac{\partial}{\partial z_2} \qquad \text{(II,20)}$$

for $t \neq a_1$, $t \neq a_2$, $t \neq a_3$

$$f_1(t) = - \frac{1}{2}\Big(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3}\Big)$$

$$f_2(t) = - \frac{1}{4} \left[\frac{b_0+b_1 t+b_2 t^2}{(t-a_1)(t-a_2)(t-a_3)^2} \right] = \\ = \frac{A_1}{t-a_1} + \frac{A_2}{t-a_2} + \frac{A_3}{(t-a_3)^2} + \frac{A_4}{t-a_3} \qquad \text{(II,21)}$$

A_1, A_2, A_3, A_4 being constants.

$$f_1^{(\varrho)}(t) = \frac{(-1)^{\varrho+1}\varrho!}{2(t-a_1)^{\varrho+1}} + \frac{(-1)^{\varrho+1}\varrho!}{2(t-a_2)^{\varrho+1}} + \frac{(-1)^{\varrho+1}\varrho!}{(t-a_3)^{\varrho+1}}$$

$$f_2^{(\varrho)}(t) = \frac{A_1(-1)^{\varrho+1}\varrho!}{(t-a_1)^{\varrho+1}} + \frac{A_2(-1)^{\varrho+1}\varrho!}{(t-a_2)^{\varrho+1}} + \frac{A_4(-1)^{\varrho+1}\varrho!}{(t-a_3)^{\varrho+1}} + \frac{A_3(-1)^{\varrho+1}(\varrho+1)!}{(t-a_3)^{\varrho+2}} \qquad (II,22)$$

With Eqs.(II,12), (II,15), (II,20), (II,21) and (II,22) Eq.(II,18) is solved and the Wangerin functions can be computated by Lie series. The Lie series solution (II,12) and (II,15) of Eq.(II,18) converges within a circle whose center is at $t = t_o$ and whose radius extends to the nearest singularity of the differential equation.

The general solution of Eq.(II,18) is given by

$$Z(t) = AW_1 + BW_2$$

where A and B are arbitrary constants and W_1, W_2 are Wangerin functions.

For computing the Wangerin functions by means of Lie series we refer to the treatment under type X in this work.

III Conclusion

In earlier reports under the contract No.NGR 52-046-001 and in this work we have investigated ordinary differential equations which result from a separation of Helmholtz and Laplace equation in various coordinate systems.

The solution functions:

Weber functions, Bessel functions, Baer functions, Mathieu functions, Legendre functions, Lamé functions, Wangerin functions and Heine functions were represented by Lie series. For computing the Weber functions, the Bessel functions and the Mathieu functions codes were written. Numerical results were published in Rep.5, Rep.6, contract NGR-52-046-001 and the monograph entitled: The Solution of Ordinary Differential Equation by Means of Lie Series, by F.Cap, W.Groebner, D.Floriani, A.Schett and J.Weil, published by NASA.

APPENDIX

Here we show that for special differential equations the series $\sum_{0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_0$ breaks off, i.e., the solution is represented by polynomials. As a special example we investigate Legendre polynomials.

Legendre Polynomials

We consider the differential equation which appears in the separation equations of spherical polar coordinates, after splitting up the singularities.

$$(t^2-1)Z'' + 2tZ' - n(n+1)Z = 0 \qquad (A,1)$$

$$Z(t) = \sum_{0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z$$

$$Z'(t) = \sum_{0}^{\infty} \frac{t^{\nu-1}}{(\nu)!} \nu D^\nu z \qquad (A,2)$$

$$Z''(t) = \sum_{0}^{\infty} \frac{t^{\nu-2}}{\nu!} \nu(\nu-1) D^\nu z$$

Inserting Eq.(A,2) in Eq.(A,1) one obtains:

$$(t^2-1)\sum_0^\infty \frac{t^{\nu-2}}{\nu!}\nu(\nu-1)\,D^\nu z + 2t\sum_0^\infty \frac{t^{\nu-1}}{\nu!}\nu\,.$$

$$.\ D^\nu z - n(n+1)\sum_0^\infty \frac{t^\nu}{\nu!}\,D^\nu z = 0$$

so that

$$\sum_0^\infty \frac{t^\nu}{\nu!}\nu(\nu-1)\,D^\nu z - \sum_0^\infty \frac{t^{\nu-2}}{\nu!}\nu(\nu-1)\,D^\nu z +$$

$$+\ 2\sum_0^\infty \frac{t^\nu}{\nu!}\nu\,D^\nu z - n(n+1)\sum_0^\infty \frac{t^\nu}{\nu!}\,D^\nu z = 0$$

if $\nu - 2 = \mu$ one obtains

$$\sum_0^\infty \frac{t^\nu}{\nu!}\Big\{\nu(\nu-1)\,D^\nu z - \frac{(\nu+2)(\nu+1)}{(\nu+2)(\nu+1)}\,D^{\nu+2}z +$$

$$+\ 2\nu D^\nu z - n(n+1)\,D^\nu z\Big\} = 0 \tag{A,3}$$

Necessary and sufficient that Eq.(A,3) is valid is the relation

$$D^\nu z\,\big\{(\nu-1)\nu + 2\nu - n(n+1)\big\} - D^{\nu+2}z = 0 \tag{A,4}$$

or

$$D^{\nu+2}z = D^\nu z\,\big\{(\nu-1)\nu + 2\nu - n(n+1)\big\} \tag{A,5}$$

If $D^0 z$ and $D^1 z$ are given one can calculate all $D^\nu z$ by Eq. (A,5).

For $\nu = n$ it follows

$$D^{n+2}z = (n^2 - n + 2n - n^2 - n)D^n z = 0$$

This means, the series $\sum_0^{\infty} \frac{t^\nu}{\nu!} D^\nu z$ breaks off, in other words we have polynomials.

It is well known, that Eq.(A,1) is solved by Legendre polynomials P_n, i.e., the relation is valid:

$$Z(t) = \sum_0^n \frac{t^\nu}{\nu!} D^\nu z = P_n$$

For computing P_n by the series $\sum_0^n \frac{t^\nu}{\nu!} D^\nu z$ we need the initial values

$$Z(t = t_0) \text{ and } Z'(t = t_0) \text{ or}$$

$$P_n(t = t_0) \text{ and } P'_n(t = t_0).$$

In analogy one can obtain other polynomials.

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